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APPROXIMATE AND STABILITY SOLUTION FOR NON-LINEAR SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE WITH BOUNDARY CONDITIONS

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Abstract

In this paper, we investigate the approximation and stability solutions of non-linear systems of integro-differential equations of Volterra type with boundary conditions, by using the numerical-analytic method which were introduced by Samoilenko. The study of such integro-differential equations leads to extend the results obtained by Butris for changing the system of non-linear integro-differential equations of Volterra type to the system of non-linear integro-differential equations of the Volterra type with boundary conditions. Theorems on a solutions are established under some necessary and sufficient conditions on compact spaces.

1.0 INTRODUCTION

A boundary value problem consists of a integro-differential equation on a given interval and an explicit condition that the solution must satisfy at one or several points. The simplest instance of such explicit conditions is when they are all specified at one initial point. The solution of integro-differential equations may be generally specified at more than one point. Often there are two points, which correspond physically to the boundaries of some region, so that it is a two–point boundary value problem [6,10,12].

The theory of integro-differential differential equations has been of great interest for many years. It plays an important role in different subjects, such as physics, biology, chemistry, etc, and the study of periodic solutions for non-linear system of differential equations with boundary. Table titles are written in Times New Roman 10-point, placed above the table, without ending the dot. The table should not be decapitated, unless it is not possible to be typed in a single page. On the next page the table lists the table numbers and is written an Advanced word without title. Charts, graphs, maps, photos, all called images. The title of the picture writes in Times New Roman 10 point, just below the image, without ending by the dot. The picture description is written in an empty space on the same page. The scale and units on the graph should be as clear as possible. Each table and picture should be referenced in the paper.

conditions and boundary integral conditions is a very important branch in the integro-differential equation theory [6,9,12,14,24]. Many results about the existence and approximation of periodic solutions for system of non–linear integro- differential equations have been obtained by the numerical analytic methods that were proposed by Samoilenko [19,20,21] which had been later applied in many studies [1, 2, 4, 7, 8, 22, 24].

The so-called numerical–analytic method for investigating a periodic solution , is widely used for studying solvability of non-linear boundary value problems and constructing approximate solutions [3, 19, 20], and it is convenient for finding harmonic oscillations arising in various systems described by ordinary differential equations, differential equations with retarded argument and with impulsive action, integro–differential equations, partial differential equations and differential equations with boundary conditions and boundary integral conditions [5, 10, 11, 13, 16, 17].

Butris [5] used the same method which were introduced by Samolinko [21] for studying the periodic solution for non-linear system of Integro-differential equations of Volterra Type which the following form:

$$\frac{dx(t)}{dt} = (A + B(t))x(t) + f(t, x(t), \sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}(s)ds)^{i}), \qquad (1.1)$$

i= 1, 2, 3,…,n

where $x \in D \subseteq R^n$, D is a closed and bounded domain.

Let the vector function $f(t,x,y) = (f_1(t,x,y), f_2(t,x,y), \dots, f_n(t,x,y))$,
Is defined and continuous on the domain,

$$(t,x,y) \in R^1 \times D \times D_1 = (-\infty,\infty) \times D \times D_1 \,, \qquad (1.2)$$

where D_1 is a bounded domain subset of Euclidean space R^m .

Suppose that the function f (t, x, y) satisfies the following inequalities:

Suppose that the function f (t ,x ,y) satisfies the following inequalities:

$$\|f(t,x,y)\| \leq M_1 \qquad (1.3)$$

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| \qquad (1.4)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$ where M_1, and L_1, L_2 are positive constants.

Suppose that G(t,s) is (n × n) continuous positive matrix and periodic in t, s of period T provided that:

$$\left.\begin{aligned} &\int_{-\infty}^{t}\|G(t,s)\|\ ds \leq K, \mathrm{K} > 0 \\ &S_1 = \sum_{i=1}^{\infty} K^i\ (\delta_0\|\mathrm{A}\|Q_1 + Q_2[H\delta_0 + M] + \beta)^{i-1} \\ &S_2 = \sum_{i=1}^{\infty} iK^i\ (\delta_0\|\mathrm{A}\|Q_1 + Q_2[H\delta_0 + M] + \beta)^{i-1} \end{aligned}\right\} \qquad (1.5)$$

Where S_1 and S_2 are convergent.

$$\text{Also } \|e^{A(t-s)}\| \leq Q_1 < \infty \,, \quad \|x_0\| = \delta_0 \;, \quad \|B(t)\| \leq H \;, N_1 = \frac{\|A\|T}{e^{\|A\|T} - \|E\|} \,,$$

$$Q_2 = Q_1(1 + N_1\, Q_1{}^2) \quad \text{and } \beta = \left\|\frac{A}{e^{AT} - E}[d_1 B_1^{-1} - (A_1 B_1^{-1} + E e^{AT})x_0]\right\|.$$

Lemma 1.2. Let the vector function $f\ (t,\, x)$ is defined and continuous on the interval [0,T]. Then the following inequality:

$$\left\| \int_0^t \left(f(s,x(s)) - \frac{1}{T}\int_0^T f(s,x(s))ds\right)ds \right\| \leq \alpha(t)\, M \tag{1.6}$$

is holds, where $\alpha(t) = 2t\left(1 - \frac{t}{T}\right)$ and $M = \max_{t\in[0,T]} |f(s,x)|$.

(for the proof see [21]).

In this work, we prove the existence and approximation of periodic solution for non-liniear system of integro-differential equations (1.1) with boundary condition:

$$A_1 x(0) + B_1 x(T) = d_1 \quad , \det B_1 \neq 0 \quad , \tag{1.7}$$

where $A_1 = (A_1 ij)$, and $B_1 = (B_1 ij)$ are $(n \times n)$ positive constant matrices and $d_1 = (d_{1_1}, d_{1_2}, \dots, d_{1_n})$ is a point in R^n .

Now, we difine the non-empty sets as follows:

$$\left.\begin{array}{l} D_\beta = D - (B_1(t)Q_1[H\delta_0 + M_1] + \beta T) \\ \\ D_{1\beta} = D_1 - (\delta_0\|A\|Q_1 + Q_2[H\delta_0 + M_1] + \beta\,) \end{array}\right\} \tag{1.8}$$

and

$$U = [(H + L_1)B_1(t)Q_1 + L_2 S_2 Q_2] < 1 \tag{1.9}$$

Also, we define the sequence of functions on the domain (1.2) as follows:

$$x_{m+1}(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}\,[B(s)x_m(s,x_0)$$

$$+ f(s, x_m(s,x_0)), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)$$

$$-\frac{A}{e^{AT} - E}\int_0^T e^{A(T-s)}\,[B(s)x_m(s,x_0) + f(s,x_m(s,x_0)), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]ds]ds$$

$$+\frac{At}{e^{AT-E}}[d_1 B_1^{-1} - (A_1 B_1^{-1} + E e^{AT})x_0] \quad , \tag{1.10}$$

with

$$x_0(t,x_0) = x_0 e^{At} \quad , \text{m}= 0\,,\, 1,\, 2\,,\dots$$

2.0 THEORETICAL

2.1. APPROXIMATE SOLUTION OF INTEGRO-DIFFERENTIAL EQUATION (1.1) WITH BOUNDARY CONDITIONS (1.7)

In this section, we investigate the approximation solution of the system (1.1) with boundary conditions (1.7).

Theorem 4.2.1. If the system (1.1) with boundary conditions (1.7) satisfy all assumptions and conditions of a above, then the sequence of functions (1.9) which is continuous in t is uniformly convergent as m → ∞ on the domain:

$$(t, x_0) \in R^1 \times D_\beta = (-\infty, \infty) \times D_\beta \tag{2.1}$$

to the limit function $x_{\infty}(t, x_0)$ defined in the domain (2.1), which is continuous in t satisfying the system of integral equations:

$$x(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)} [B(s)x(s,x_0) + f(s, x(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)} [B(s)x(s,x_0) + f(s, x(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds]ds$$

$$+\frac{At}{e^{AT}-E}[d_1 B_1^{-1} - (A_1 B_1^{-1} + E e^{AT})x_0] \quad , \qquad (2.2)$$

wich is a unique solution on the domain (2.1) provided that:

$$\|x_{\infty}(t,x_0) - x_0\| \le B_1(t)Q_1[H\delta_0 + M_1] + \beta T \qquad (2.3)$$

and

$$\|x_{\infty}(t,x_0) - x_m(t,x_0)\| \le B_1(t)Q_1U_1(1-U)^{-1}U^{m-1} \quad , \qquad (2.4)$$

for all $m \ge 1$, $x_0 \in D_\beta$ and $t \in R^1$.

Proof. Setting m=0 in the sequance of function (1.9) and using lemma 1.1, we have:-

$$\|x_1(t,x_0) - x_0\| \le \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right] \int_0^t \|e^{A(t-s)}\| \, [\|B(s)\|\|x_0\| + \|f(s,x_0,0)\|]ds$$

$$+\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right) \int_t^T \|e^{A(t-s)}\| \, [\|B(s)\|\|x_0\| + \|f(s,x_0,0)\|]ds$$

$$+\left\|\frac{At}{e^{AT}-E}[d_1 B_1^{-1} - (A_1 B_1^{-1} + E e^{AT})x_0]\right\|$$

$$\|x_1(t,x_0) - x_0\|$$

$$\le \left[\frac{t(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|E\|) + T(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|}\right] Q_1[H\delta_0 + M] + \beta T$$

$$\le B_1(t)Q_1[H\delta_0 + M_1] + \beta T$$

where $B_1(t) = \left[\frac{t(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|E\|) + T(e^{\|A\|T} - e^{\|A\|(T-t)})}{e^{\|A\|T} - \|E\|}\right]$,

Hence,

$$\|x_1(t,x_0) - x_0\| \le B_1(t)Q_1[H\delta_0 + M_1] + \beta T \qquad (2.5)$$

i.e. $x_1(t,x_0) \in D$, for all $x_0 \in D_\beta$ and $t \in R^1$.

Moreover, on differentiating $x_1(t,x_0)$, we find that:

$$\|\dot{x}_1(t,x_0)\| = \| x_0Ae^{At} + e^{A(t-s)}[B(t)x_0 + f(t,x_0,0) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}\,[B(s)x_0$$

$$+f(s,x_0,0)]ds] + \frac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0]\,\|$$

$$\le \|x_0\|\|A\|\|e^{At}\| + \|e^{A(t-s)}\|[\|B(t)\|\|x_0\| + \|f(t,x_0,0)\| +$$

$$\frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T \|e^{A(T-s)}\|\,[\|B(s)\|\|x_0\|$$

$$+\|f(s,x_0,0)\|]ds] + \|\,\frac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0]\quad\|$$

$$\le \delta_0\|A\|Q_1 + Q_1[H\delta_0 + M_1 + \frac{\|A\|}{e^{\|A\|T}-\|E\|}\int_0^T Q_1\,[H\delta_0 + M_1]ds] + \beta$$

$$\le \delta_0\|A\|Q_1 + Q_1[H\delta_0 + M_1] + \frac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|}[H\delta_0 + M_1]] + \beta$$

$$\le \delta_0\|A\|Q_1 + (Q_1 + \frac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|})[H\delta_0 + M_1] + \beta$$

$$\le \delta_0\|A\|Q_1 + Q_2[H\delta_0 + M_1] + \beta \quad ,$$

And hance,

$$\|\dot{x}_1(t,x_0)\| \le \delta_0\|A\|Q_1 + Q_2[H\delta_0 + M_1] + \beta \quad , \qquad (2.6)$$

i.e. $\dot{x}_1(t,x_0) \in D_1$, for all $x_0 \in D_\beta$ and $t \in R^1$.

Thus, by induction, we prove that :

$$\left.\begin{aligned} &\|x_m(t,x_0) - x_0\| \le B_1(t)Q_1[H\delta_0 + M_1] + \beta T\,, \\ &\|\dot{x}_m(t,x_0)\| \le \delta_0\|A\|Q_1 + Q_2[H\delta_0 + M_1] + \beta \end{aligned}\right\} \qquad (2.7)$$

i.e. $x_m(t,x_0) \in D$, $\dot{x}_m(t,x_0) \in D_1$, for all $x_0 \in D_\beta$ and $t \in R^1$,

Where

$$\dot{x}_{m+1}(t,x_0) = x_0Ae^{At} + e^{A(t-s)}[B(t)x_m(t,x_0)$$

$$+ f(t,x_m(t,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_m(s,x_0)ds)^i)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}\,[B(s)x_m(s,x_0) +$$

$$f(s,x_m(s,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]ds]$$

$$+\frac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0] \quad , \qquad (2.8)$$

For all m= 0, 1, 2, …

Next, we shall prove the sequence of function $\{x_m(t,x_0)\}_{m=0}^{\infty}$ converges uniformly to the limit funtion $x_\infty(t,x_0)$ on the domain (2.1) which is a required solution of integro-differential equation (1.1) with boundary conditions (1.7), and to do this, we need to prove the following inequality :

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le B_1(t)QU_1U^{m-1} \quad , \text{ m= 1, 2, 3,\dots} \tag{2.9}$$

where

$U=[(H+L_1)B_1(t)Q_1+L_2S_2Q_2]$, $U_1=[(H+L_1)(B_1(t)Q_1[H\delta_0+M_1]+\beta T)+L_2S_1(\delta_0\|\mathrm{A}\|Q_2+Q_2[H\delta_0+M_1]+\beta)]$, .

From the sequence of functions (1.9), we have:

$$\|x_2(t,x_0)-x_1(t,x_0)\| = \|x_0e^{At}+\int_0^t e^{A(t-s)}\,[B(s)x_1(s,x_0)$$
$$+f(s,x_1(s,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}_1\,(\tau,x_0)d\tau)^i)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}\,[B(s)x_1(s,x_0)$$
$$+f(s,x_1(s,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}_1\,(\tau,x_0)d\tau)^i)]ds]ds+\frac{At}{e^{AT}-E}[d_1B_1^{-1}-(A_1B_1^{-1}+$$
$$Ee^{AT})x_0]$$
$$-x_0e^{At}-\int_0^t e^{A(t-s)}\,[B(s)x_0+f(s,x_0,0)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}\,[B(s)x_0$$
$$+f(s,x_0,0)]ds]ds$$
$$-\frac{At}{e^{AT}-E}[d_1B_1^{-1}-(A_1B_1^{-1}+Ee^{AT})x_0]\,\| \quad ,$$

By using lemma 1.1, we get:

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le \left[\|E\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t\|e^{A(t-s)}\|\,[\|B(s)\|\|x_1(s,x_0)-x_0\|$$
$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\textstyle\sum_{i=1}^{\infty}K^i\,((\delta_0\|A\|Q_1+Q_2[H\delta_0+M_1]+$$
$$\beta)^{i-1})\|\dot{x}_1(s,x_0)\|]ds$$
$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T\|e^{A(t-s)}\|\,[\|B(s)\|\|x_1(s,x_0)-x_0\|$$
$$+L_1\|x_1(s,x_0)-x_0\|+L_2(\textstyle\sum_{i=1}^{\infty}K^i\,((\delta_0\|A\|Q_1+Q_2[H\delta_0+M_1]+$$
$$\beta)^{i-1})\|\dot{x}_1(s,x_0)\|]ds$$

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le \left[\|E\|-\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right]\int_0^t Q_1\,[H\|x_1(s,x_0)-x_0\|+$$
$$L_1\|x_1(s,x_0)-x_0\|$$
$$+L_2S_1\|\dot{x}_1(s,x_0)\|]ds+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T Q_1\,[H\|x_1(s,x_0)-x_0\|$$
$$+L_1\|x_1(s,x_0)-x_0\|+L_2S_1\|\dot{x}_1(s,x_0)\|]ds$$

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le B_1(t)Q_1[(H+L_1)(\,B_1(t)Q[H\delta_0+M_1]+\beta T)+L_2S_1(\delta_0\|\mathrm{A}\|Q_1+$$
$$Q_2[H\delta_0+M_1]+\beta)] \;,$$

let $U_1=[(H+L_1)(\,B_1(t)Q_1[H\delta_0+M_1]+\beta T)+L_2S_1\,(\delta_0\|\,A\|Q_1+Q_2[H\delta_0+M_1]+\beta)]$

So

$\|x_2(t,x_0) - x_1(t,x_0)\| \leq B_1(t)Q_1U_1$.

Hence, the inequality (2.9) is true for m= 1 .

Suppose that the inequality (2.8) is true for $p \in Z^+$, i.e.

$$\left.\begin{aligned} &\|x_{p+1}(t,x_0) - x_p(t,x_0)\| \leq B_1(t)Q_1U_1U^{p-1} \\ &\|\dot{x}_{p+1}(t,x_0) - \dot{x}_p(t,x_0)\| \leq Q_2U_1U^{p-1} \end{aligned}\right\} \qquad (2.10)$$

Then from the sequence of functions (1.7) again and using lemma 1.1 ,we find that:-

$$\|x_{p+2}(t,x_0) - x_{p+1}(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right] \int_0^t \|e^{A(t-s)}\| \, [\|B(s)\|\|x_{p+1}(s,x_0) -$$

$$x_p(s,x_0)\|$$

$$+L_1\|x_{p+1}(s,x_0) - x_p(s,x_0)\| + L_2(\textstyle\sum_{i=1}^{\infty} iK^i \ (\delta_0NQ_1 + Q_2[H\delta_0 + M_1] +$$

$$\beta)^{i-1})\|\dot{x}_{p+1}(s,x_0) - \dot{x}_p(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T \|e^{A(t-s)}\| \, [\|B(s)\|\|x_{p+1}(s,x_0) - x_p(s,x_0)\|$$

$$+L_1\|x_{p+1}(s,x_0) - x_p(s,x_0)\| + L_2(\textstyle\sum_{i=1}^{\infty} iK^i \ (\delta_0NQ_1 + Q_2[H\delta_0 + M_1] +$$

$$\beta)^{i-1})\|\dot{x}_{p+1}(s,x_0) - \dot{x}_p(s,x_0)\|]ds$$

$$\|x_{p+2}(t,x_0) - x_{p+1}(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right] \int_0^t Q_1 \, [H\|x_{p+1}(s,x_0) - x_p(s,x_0)\|$$

$$+L_1\|x_{p+1}(s,x_0) - x_p(s,x_0)\| + L_2S_2\|\dot{x}_{p+1}(s,x_0) - \dot{x}_p(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T Q_1 \, [H\|x_{p+1}(s,x_0) - x_p(s,x_0)\|$$

$$+L_1\|x_{p+1}(s,x_0) - x_p(s,x_0)\| + L_2S_2\|\dot{x}_{p+1}(s,x_0) - \dot{x}_p(s,x_0)\|]ds$$

$$\|x_{p+2}(t,x_0) - x_{p+1}(t,x_0)\| \leq B_1(t)Q_1[HB_1(t)Q_1U_1U^{P-1} + L_1B_1(t)Q_1U_1U^{P-1} +$$

$$L_2S_2Q_2U_1U^{P-1}]$$

$$\leq B_1(t)Q_1U_1U^{P-1}[(H+L_1)B_1(t)Q_1 + L_2S_2Q_2]$$

$$\leq B_1(t)Q_1U_1U^P,$$

and hence ,

$$\|x_{p+2}(t,x_0) - x_{p+1}(t,x_0)\| \leq B_1(t)Q_1U_1U^P \quad . \qquad (2.11)$$

Therefore, by induction the inequality (2.8) is true for m ≥ 1.

From the inequality (2.11), we get the following inequality:

$$\|x_{m+k}(t,x_0) - x_m(t,x_0)\| \le \sum_{j=0}^{k-1} B_1(t)\, Q_1 U_1 U^{m-1+j} \quad , \tag{2.12}$$

holds for all k >1 and $x_0 \in D_\beta$.

$$\sum_{j=0}^{k-1} U^{m-1+j} \le U^{m-1} \sum_{j=0}^{\infty} U^j = U^{m-1}(1-U)^{-1} \quad , \tag{2.13}$$

and

$$\lim_{m\to\infty} U^{m-1} = 0 \text{ , where } m= 1, 2, 3,\ldots, \tag{2.14}$$

Relation (2.13) and (2.14) prove the uniform convergence of the sequence of function (1.9) in the domain (2.1) as m → ∞ .

Let

$$\lim_{m\to\infty} x_m(t,x_0) = x_\infty(t,x_0) \quad . \tag{2.15}$$

Since the sequence of functions (1.9) is continuous in t , then the limiting function $x_\infty(t,x_0)$ is also continuous in t .

Moreover, by the lemma 1.1 , and the inequality (2.13) the inequalities (2.3) and (2.4) hold for all $m \ge 1$.

Using relation (2.15) , when m → ∞ , that the limiting function $x_\infty(t,x_0)$ is a solution of the integral equation (2.2).

Finally, we show that x (t, x 0) is a unique solution of the system (1.1) with boundary conditions (1.7). Assume that r (t, x 0) is another solution for the system (1.1) with boundary conditions (1.7), i.e.

$$r(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)} \left[B(s)r(s,x_0) + f(s, r(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i\right)$$

$$-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}\,[B(s)r(s,x_0) + f(s,r(s,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i)]ds]ds$$

$$+\frac{At}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0] \quad , \tag{2.16}$$

Now, we find the difference between them, we have:

$$\|x(t,x_0) - r(t,x_0)\| \le \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right] \int_0^t \|e^{A(t-s)}\| \,[\|B(s)\|\|x(s,x_0) - r(s,x_0)\|$$

$$+L_1\|x(s,x_0) - r(s,x_0)\| + L_2(\textstyle\sum_{i=1}^{\infty} iK^i\,(\delta_0 NQ_1 + Q_2[H\delta_0 + M] +$$

$$\beta)^{i-1})\|\dot{x}(s,x_0) - \dot{r}(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right) \int_t^T \|e^{A(t-s)}\| \,[\|B(s)\|\|x(s,x_0) - r(s,x_0)\|$$

$$+L_1\|x(s,x_0) - r(s,x_0)\| + L_2(\textstyle\sum_{i=1}^{\infty} iK^i\,(\delta_0\|A\|Q_1 + Q_2[H\delta_0 + M_1] +$$

$$\beta)^{i-1})\|\dot{x}(s,x_0) - \dot{r}(s,x_0)\|]ds$$

Or

$$\|x(t,x_0)-r(t,x_0)\| \leq \left[\|E\| - \left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\right] \int_0^t Q_1\,[H\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2S_2\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right) \int_t^T Q_1\,[H\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2S_2\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds$$

Thus

$$\|x(t,x_0)-r(t,x_0)\| \leq B_1(t)Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\|$$

$$+L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|] \tag{2.17}$$

Also, we find :

$$\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| = \|x_0Ae^{At} + e^{A(t-s)}[B(t)x(t,x_0)$$

$$+f(t,x(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}(s,x_0)ds)^i) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-s)}\,[B(s)x(s,x_0)$$

$$+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds] + \frac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0]$$

$$-x_0Ae^{At} - e^{A(t-s)}[B(t)r(t,x_0) + f(t,r(t,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{r}(s,x_0)ds)^i)$$

$$-\tfrac{A}{e^{AT}-E}\textstyle\int_0^T e^{A(T-s)}\,[B(s)r(s,x_0) + f(s,r(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{S} G(s,\tau)\dot{r}(\tau,x_0)d\tau)^i)]ds]$$

$$-\tfrac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0]\| \,.$$

By using inequality (2.4) , we have:

$$\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| \leq \|e^{A(t-s)}\|[\|B(t)\|\|x(t,x_0)-r(t,x_0)\| + L_1\|x(t,x_0)-r(t,x_0)\|$$

$$+L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| + \tfrac{\|A\|}{e^{\|A\|T}-\|E\|}\textstyle\int_0^T \|e^{A(T-s)}\|\,[\|B(s)\|\|x(s,x_0)-r(s,x_0)\|$$

$$+L_1\|x(s,x_0)-r(s,x_0)\| + L_2S_2\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds]$$

Or

$$\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| \leq Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|$$

$$+\tfrac{\|A\|}{e^{\|A\|T}-\|E\|}\textstyle\int_0^T Q_1\,[(H+L_1)\|x(s,x_0)-r(s,x_0)\| + L_2S_2\|\dot{x}(s,x_0)-\dot{r}(s,x_0)\|]ds]$$

$$\leq (Q_1 + \tfrac{\|A\|TQ^2}{e^{\|A\|T}-\|E\|})[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$$

and hence,

$$\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\| \le Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|], \tag{2.18}$$

From inequality (2.17) and (2.18), we obtain that:

$$\|x(t,x_0)-r(t,x_0)\| \le B_1(t)Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$$

$$\begin{aligned} &\le B_1(t)Q_1\{(H+L_1)B_1(t)Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|] \\ &\qquad +L_2S_2Q_2[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]\} \\ &\le B_1(t)Q_1((H+L_1)B_1(t)Q_1 + L_2S_2Q_2)[(H+L_1)\|x(t,x_0)-r(t,x_0)\| \\ &\qquad +L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|] \le C_2U , \end{aligned}$$

Hence ,

$\|x(t,x_0)-r(t,x_0)\| \le C_2U$,

where $C_2 = B_1(t)Q_1[(H+L_1)\|x(t,x_0)-r(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{r}(t,x_0)\|]$

so, by induction, we obtain :

$$\|x(t,x_0)-r(t,x_0)\| \le C_2U^m \quad . \tag{2.19}$$

From inequality (2.19) and by using (2.4), when $m \to \infty$, $U^m \to 0$,we have:

$\|x(t,x_0)-r(t,x_0)\| = 0$,

thus, $x(t,x_0) = r(t,x_0)$, i.e.

$x(t,x_0)$ is a unique solution of the system (1.1) with boundary conditions (1.7) for all $t \in R^1$, $x_0 \in D_\beta$.

3.0 METHODOLOGY

Existance of Δ (0, x_0) For The System (1.1) With Boundary Conditions (1.7).

The existance of Δ (0, x_0) for thesSystem (1.1) with boundary conditions (1.7). is uniquely connected by the following function:

$$\begin{aligned} \Delta(0,x_0) = &\frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}\,[B(t)x_\infty(t,x_0) + \\ &f(t,x_\infty(t,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}_\infty(s,x_0)ds)^i)]dt \\ &+\frac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0] , \end{aligned} \tag{3.1}$$

Where x_∞ (t, x_0) is the limiting function of the sequence of functions (1.9).

This function is an approximation determined from the following sequence of functions:

$$\Delta_m(0,x_0) = \frac{A}{e^{AT}-E}\int_0^T e^{A(T-t)}\,[B(t)x_m(t,x_0)$$

$$+ f(t,x_m(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_m(s,x_0)ds)^i)]dt$$

$$+\tfrac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} + Ee^{AT})x_0]\,, \qquad (3.2)$$

where m= 0, 1, 2, 3,…

Theorem 3.1. If the hypotheses and all conditions of theorem 2.1 are holds, then th following inequality is satisfied :

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le N_1N_2Q^2(H+L_1)B_1(t)U_1(1-U)^{-1}U^{m-1} = \gamma_m \qquad (3.3)$$

where $N_2 = [1 + L_2S_2Q_1(1-Q_1L_2S_2)^{-1}]\,,\ Q_2 = Q_1(1+N_1{Q_1}^2)$

Proof. From (3.1) and (3.2) ,we have :

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le \frac{\|A\|}{e^{\|A\|T} - \|E\|}\int_0^T \|e^{A(T-t)}\|\,[\|B(t)\|\|x_\infty(t,x_0) - x_m(t,x_0)\|$$

$$+L_1\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2(\textstyle\sum_{i=1}^{\infty} iK^i\,(\delta_0\|A\|Q + Q_1[H\delta_0 + M] +$$
$$\beta)^{i-1})\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]dt$$

or

$$\|\Delta(x_0) - \Delta_m(x_0)\| \le \tfrac{\|A\|}{e^{\|A\|T}-\|E\|}\textstyle\int_0^T Q_1\,[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\|$$
$$+L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]dt$$

$$\le (\tfrac{\|A\|T}{e^{\|A\|T}-\|E\|})Q_1[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]$$

$$\le N_1Q_1[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]\,,$$

and hence,

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le N_1Q[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) -$$
$$\dot{x}_m(t,x_0)\|] \qquad (3.4)$$

Taking

$$\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\| \le Q_1[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) -$$
$$\dot{x}_m(t,x_0)\|$$

$$+N_1Q_1[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]]$$

$$\le\ Q_2\ [(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]$$

$$\le Q_2[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]$$

and hence,

$$\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\| \le Q_2[(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| + L_2S_2\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\|]$$

From the last inequality, we have:

$$\|\dot{x}_\infty(t,x_0) - \dot{x}_m(t,x_0)\| \le (1 - Q_2L_2S_2)^{-1}Q_2(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| \qquad (3.5)$$

By substituting inequality (3.5) in (3.4) , we obtain:

$$\begin{aligned}\|\Delta(0,x_0) - \Delta_m(0,x_0)\| &\le N_1Q_1(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| \\ &\quad + N_1Q_1(H+L_1)L_2S_2\,Q_2(1-Q_2L_2S_2)^{-1}\|x_\infty(t,x_0) - x_m(t,x_0)\| \\ &\le N_1Q_1(H+L_1)[1 + L_2S_2Q_2(1-Q_2L_2S_2)^{-1}]\|x_\infty(t,x_0) - x_m(t,x_0)\| \\ &\le N_1N_2Q_1(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| \ ,\end{aligned}$$

So that

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le N_1N_2Q_1(H+L_1)\|x_\infty(t,x_0) - x_m(t,x_0)\| \ .$$

Using inequality (2.4), we get:

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le N_1N_2\,{Q_1}^2(H+L_1)B_1(t)U_1(1-U)^{-1}U^{m-1} = \gamma_m$$

i.e. the inequality (3.3) will be satisfied for all $m \ge 1$.

Theorem 3.2. Let the system (1.1) with boundary conditions (1.7) be defined in the interval [c, d] on R^1 Assume that the sequence of functions (1.10) satisfies the following inequalities:

$$\left.\begin{array}{l} min\,\Delta_m\,(0,x_0) \le -\gamma_m \ , \\ \\ max\,\Delta_m\,(x0,_0) \le \gamma_m \end{array}\right\} \qquad (3.6)$$

where $x_0 \in [c + B_1(t)Q_1[H\delta_0 + M_1], d - B_1(t)Q_1[H\delta_0 + M_1]$ and

$$\gamma_m = \|N_1N_2{Q_1}^2(H+L_1)B_1(t)U_1(1-U)^{-1}U^{m-1}\| \ .$$

Then , the system (1.1) with boundary conditions (1.7) has a solution $x = x(t,x_0)$ for $x_0 \in [c + B_1(t)Q_1[H\delta_0 + M_1], d - B_1(t)Q_1[H\delta_0 + M_1]$.

Proof. Let x_1, x_2 be any two points in the interval $[c + B_1(t)Q_1[H\delta_0 + M_1], d - B_1(t)Q_1[H\delta_0 + M_1]$ such that :

$$\left.\begin{aligned} \Delta_m(0,x_1) &= \min \Delta_m(0,x_0) \ , \\ \Delta_m(0,x_2) &= \max \Delta_m(0,x_0) \end{aligned}\right\} \qquad (3.7)$$

where $x_0 \in [c + B_1(t)Q_1[H\delta_0 + M_1], d - B_1(t)Q_1[H\delta_0 + M_1]$

By using the inequalities (3.3) and (3.6) ,we have :

$$\left.\begin{aligned} \Delta(0,x_1) &= \Delta_m(0,x_1) + (\Delta(0,x_1) - \Delta_m(0,x_1)) \\ \Delta(0,x_2) &= \Delta_m(0,x_2) + (\Delta(0,x_2) - \Delta_m(0,x_2)) \end{aligned}\right\} \begin{aligned} &\le 0, \\ &\ge 0 \end{aligned} \qquad (3.8)$$

It follows from the inequalities (3.8) and the continuity of the function $\Delta(0,x_0)$ that there exists an isolated singular point $x_\infty = x_0$, $x_\infty \in [x_1, x_2]$, such that $\Delta(0,x_0) = 0$. This means that the system (1.1) with boundary conditions (1.7) has a solution $x(t,x_0)$ for which $x_0 \in [c + B_1(t)Q[H\delta_0 + M], d - B_1(t)Q[H\delta_0 + M]$

Remark 3.1[21]. Theorem 3.2 is proved when x_0 is a scalar singular point which should be isolated , thus we have :

4.0 RESULANTS

Stability Solution Of Integro-Differential Equation (1.1) With Boundary Conditions (1.7)

In this section, we investigate the stability solution of the system (1.1) with boundary conditions (1.7). Theorem 4.1. Suppose that the function Δ (0, x) as defined in (3.1) is given, then we get the following inequalities:

$$\|\Delta(0,x_0)\| \le M_2 \qquad (4.1)$$

where

$$M_2 = N_1Q(H\delta_0Q_1M_3 + B_1(t)Q_1HMM_3 + \beta HTM_3 + M_1) + \beta$$

where $M_3 = (1 - \|B_1(t)\|Q_1H)^{-1}$

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \le [F_2F_4\|(B_1(t)\|E_2F_1(N_1Q_1 + \beta_1) + (Q_1 + T\beta_1)) + F_3]\|x_0^1 - x_0^2\| \qquad (4.2)$$

hold for all $x_0, x_0^1, x_0^2 \in D_\beta$, where

$$\beta_1 = \left\|\tfrac{A}{e^{AT}-E}[A_1B_1{}^{-1} + Ee^{AT}]\right\| \quad , \quad E_1 = Q_1(H + L_1) \text{ and } E_2 = Q_1L_2S_2$$

Proof. From the equation (3.1) , we have :

$$\|\Delta(0,x_0)\| \le \frac{\|A\|}{e^{\|A\|T} - \|E\|}\int_0^T \|e^{A(T-t)}\| \, [\|B(t)\|\|x_\infty(t,x_0)\|$$

$$+ \left\|f(t,x_\infty(t,x_0), \textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_\infty(s,x_0)ds)^i)\right\|]dt + \left\|\tfrac{A}{e^{AT}-E}[d_1B_1^{-1} - (A_1B_1^{-1} +\right.$$

$$\left.Ee^{AT})x_0]\right\|$$

So, we have

$$\|\Delta(0,x_0)\| \leq \frac{\|A\|T}{e^{\|A\|T} - \|E\|} Q_1[H\|x_\infty(t,x_0)\| + M_1] + \beta$$

$$\|\Delta(0,x_0)\| \leq \frac{\|A\|T}{e^{\|A\|T} - \|E\|} Q_1 H\|x_\infty(t,x_0)\| + \frac{\|A\|T}{e^{\|A\|T} - \|E\|} Q_1 M_1 + \beta \qquad (4.3)$$

Since the function $x_\infty(t,x_0)$ satisfies the integral equation (2.2), then we find that:

$$\|x_\infty(t,x_0)\| = \|x_0\|\|e^{At}\| + \| \int_0^t e^{A(t-s)}[B(s)x_\infty(s,x_0)$$

$$+f(s,x_\infty(s,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_\infty(\tau,x_0)d\tau)^i) - \frac{A}{e^{AT-E}}\int_0^T e^{A(T-s)}[B(s)x_\infty(s,x_0)$$

$$+f(s,x_\infty(s,x_0),\textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_\infty(\tau,x_0)d\tau)^i)]ds]ds + \frac{At}{e^{AT-E}}[d_1B_1^{-1} - (A_1B_1^{-1} +$$

$$Ee^{AT})x_0]\|$$

Now, by using lemma 1.1, we get:

$$\|x_\infty(t,x_0)\| = \delta_0 Q_1 + \|B(t)\|Q_1[H\|x_\infty(s,x_0)\| + M_1] + \beta T$$

so,

$$\|x_\infty(t,x_0)\| = \delta_0 Q M_1 + \|B(t)\|Q M_1 M_3 + \beta T M_3 \qquad (4.4)$$

Substituting inequality (4.3) in (4.2), we get the inequality (3.7).
Also, by using equation (3.1), we have:

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \leq \frac{\|A\|T}{e^{\|A\|T} - \|E\|} Q[H\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\| + L_1\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\|$$

$$+L_2S_6\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\|] + \left\|\frac{A}{e^{AT-E}}[A_1{B_1}^{-1} + Ee^{AT}]\right\| \|x_0^1 - x_0^2\|$$

so,

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \leq N_1E_1\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\| + N_1E_4\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\|$$
$$+\beta_1\|x_0^1 - x_0^2\| \qquad (4.5)$$

Now, we to find that :

$$\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\| \leq \|x_0^1 - x_0^2\|\|A\|\|e^{At}\| + \|e^{A(t-s)}\|[\|B(t)\|\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\|$$

$$+L_1\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\| + L_2S_2\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\|]$$

$$+\frac{\|A\|}{e^{\|A\|T} - \|E\|}\int_0^T \|e^{A(T-s)}\| \, [\|B(s)\|\|x_\infty(s,x_0^1) - x_\infty(s,x_0^2)\|$$

$$+L_1\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\| + L_2S_2\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\|]ds$$

$$+\left\|\frac{A}{e^{AT-E}}[A_1{B_1}^{-1} + Ee^{AT}]\right\| \|x_0^1 - x_0^2\|$$

$$\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\| \leq \|x_0^1 - x_0^2\|\|A\|Q_1 + Q_1[(H + L_1)\|x_\infty(t,x_0^1) - x_\infty(t,x_0^2)\|$$

$$+L_2S_2\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\| + N_1Q_1[(H + L_1)\|x_\infty(s,x_0^1) - x_\infty(s,x_0^2)\|$$

$$+L_2S_2\|\dot{x}_\infty(t,x_0^1) - \dot{x}_\infty(t,x_0^2)\|]] + \beta_1\|x_0^1 - x_0^2\|$$

$$\leq \|x_0^1 - x_0^2\|(\|A\|Q_1 + \beta_1) + Q_2[(H + L_1)\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\|$$

$$+L_2 S_2\|\dot{x}_\infty(t, x_0^1) - \dot{x}_\infty(t, x_0^2)\|]$$

and hence ,

$$\|\dot{x}_\infty(t, x_0^1) - \dot{x}_\infty(t, x_0^2)\| \leq \|x_0^1 - x_0^2\|(\|A\|Q_1 + \beta_1)(1 - Q_2 L_2 S_2)^{-1}$$
$$+Q_2(H + L_1)(1 - Q_2 L_2 S_2)^{-1}\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\|$$
$$\leq F_1(\|A\|Q_1 + \beta_1)\|x_0^1 - x_0^2\| + F_1 Q_2(H + L_1)\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| \tag{4.6}$$

where

$$F_1 = (1 - Q_2 L_2 S_2)^{-1}$$

By substituting inequality (4.6) in (4.5) , we obtain :

$$\|\Delta(0, x_0^1) - \Delta(0, x_0^2)\| \leq N_2 E_1\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + N_2 E_2 F_1(\|A\|Q_1 Q + \beta_1)\|x_0^1 - x_0^2\|$$
$$+N_2\ E_2 F_1 Q_2\,(H + L_1)\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + \beta_1\|x_0^1 - x_0^2\|$$
$$\leq N_2\,(E_1 +\ E_2 2F_1(H + L_1))\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\|$$
$$+(N_2\,E_2 F_1\,(N_2 Q_1 + \beta_1) + \beta_1)\|x_0^1 - x_0^2\|$$

Putting

$$F_2 = N_2(E_1 + E_2 Q_1 F_1(H + L_1)) \quad \text{and} \quad F_3 = (N_2 E_2 F_1(N_1 Q + \beta_1) + \beta_1)$$

so, the last inequality becomes :

$$\|\Delta(0, x_0^1) - \Delta(0, x_0^2)\| \leq F_2\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| + F_3\|x_0^1 - x_0^2\| \tag{4.7}$$

where $x_\infty(t, x_0^1)$ and $x_\infty(t, x_0^2)$ are the solutions of the integral equation :

$$x(t, x_0^k) = x_0^k e^{At} + \int_0^t e^{A(t-s)}\,[B(s)x(s, x_0^k) + f(s, x(s, x_0^k), \textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau, x_0^k)d\tau)^i)$$
$$-\frac{A}{e^{AT-E}}\int_0^T e^{A(T-s)}\,[B(s)x(s, x_0^k) + f(s, x(s, x_0^k), \textstyle\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau, x_0^k)d\tau)^i)]ds]ds$$
$$+\frac{At}{e^{AT-E}}[d_1 B_1^{-1} - (A_1 B_1^{-1} + Ee^{AT})x_0^k] \tag{4.8}$$

With

$x_0^k(t, x_0) = x_0^k$, where k =1, 2 .

From the equation (4.8) and using lemma 1.1, we have:

$$\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| \le \|x_0^1 - x_0^2\| \|e^{At}\|$$

$$+ \left[\|E\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\right] \int_0^t \|e^{A(t-s)}\| \, [\|B(s)\| \|x_\infty(s, x_0^1) -$$

$$x_\infty(s, x_0^2)\|$$

$$+ L_1 \|x_\infty(s, x_0^1) - x_\infty(s, x_0^2)\| + L_2 S_2 \|\dot{x}_\infty(s, x_0^1) - \dot{x}_\infty(s, x_0^2)\|] ds$$

$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right) \int_t^T \|e^{A(t-s)}\| \, [\|B(s)\| \|x_\infty(s, x_0^1) - x_\infty(s, x_0^2)\|$$

$$+ L_1 \|x_\infty(s, x_0^1) - x_\infty(s, x_0^2)\| + L_2 S_2 \|\dot{x}_\infty(s, x_0^1) - \dot{x}_\infty(s, x_0^2)\|] ds$$

$$+ T \left\| \frac{A}{e^{AT} - E} [A_1 {B_1}^{-1} + E e^{AT}] \right\| \|x_0^1 - x_0^2\|$$

So that:

$$\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| \le \|x_0^1 - x_0^2\| (Q_1 + T\beta_1) + \frac{T}{2} Q_1 (H + L_1) \|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\|$$

$$+ \frac{T}{2} Q_1 L_2 S_2 \|\dot{x}_\infty(s, x_0^1) - \dot{x}_\infty(s, x_0^2)\| \qquad (4.9)$$

Now, by substituting inequality (4.9) in (4.7), we get:

$$\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| \le (\|B(t)\| L_2 S_2 F_1 Q_2 E_1 + \|B(t)\| E_1) \|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\|$$

$$+ (\|B(t)\| E_2 F_1 (\|A\| Q_1 + \beta_1) + (Q_1 + T\beta_1)) \|x_0^1 - x_0^2\|$$

So

$$\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| \le (1 - (\|B(t)\| L_2 S_2 F_1 Q_2 E_1 + \|B(t)\| E_1))^{-1}$$

$$(\|B(t)\| E_2 F_1 (\|A\| Q_1 + \beta_1) + (Q_1 + T\beta_1)) \|x_0^1 - x_0^2\|$$

$$\|x_\infty(t, x_0^1) - x_\infty(t, x_0^2)\| \le F_4 (\|B(t)\| E_2 F_1 (\|A\| Q_1 + \beta_1) + (Q_1 + T\beta_1)) \|x_0^1 - x_0^2\| \qquad (4.10)$$

where

$$F_4 = (1 - (\|B(t)\| L_2 S_2 F_1 Q_2 E_1 + \frac{T}{2} E_1))^{-1}$$

Also, substituting inequality (4.10) in (4.7), we get the inequality (4.2).

Remark 4.1 [8]. Theorem 4.1, confirms the stability of the solution for the system (1.1) with boundary conditions (1.7), that is when a slight change happens in the point x $_0$, then a slight change will happen in the function $x = x(t, x_0)$.

5.0 CONCLUSION

Through the use of Samoilenko's numerical-analytic method of the periodic solutions of non-linear differential equations which are given by [21], the present study reached the following results, the existence of function *Δ* (0, *x*$_0$), which is equal to zero, determines the initial value of the solutions of non-linear systems of integro-differential equations of Volterra type where the properties of function $\Delta(0, x_0)$ play a basic role in building a solutions of non-linear systems of integro-differential equations of Volterra type, especially in the following integro-differential equations :

$$\frac{dx(t)}{dt} = (A + B(t))x(t) + \left[f\left(t, x(t), \sum_{i=1}^{\infty}\left(\int_{-\infty}^{t} G(t,s)\dot{x}(s)ds\right)^{i}\right) - \Delta\right]$$

with boundary conditions:

$$A_1x(0) + B_1x(T) = d_1 \quad , det\, B_1 \neq 0.$$

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